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DIFFRACTION OF NORMAL MODES IN COMPOSITE AND STEPPED ELASTIC WAVEGUIDES†

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A method of calculating the diffraction of Lamb waves at the vertical junctions of elastic waveguides is proposed, the effectiveness of which is ensured by taking into account the nature of the singularity of the solution at corner points. The property of the generalized orthogonality of normal modes plays a key role. It enables the coefficients of the expansion of the wave field in the modes to be expressed in terms of the displacements and stresses along the junction line. Numerical results are presented which show how the transmission coefficients and the energy distribution depend on the height of the step and frequency for a stepped waveguide, attached to an undeformed substrate. © 1998 Elsevier Science Ltd. All rights reserved.

In addition to direct numerical schemes and asymptotic expansions, the fundamental semi-analytic method of solving problems of the diffraction of elastic waves by surface steps, the joining lines of half-strips, vertical cracks, etc. is an expansion in normal modes (the method of superposition or piecewise-homogeneous solutions) [1–6]. However, the infinite algebraic systems

$$A\mathbf{t} = \mathbf{f} \tag{1}$$

which arise here with respect to the unknown coefficients of the expansion $\mathbf{t} = \{\mathbf{t}_1, \mathbf{t}_2, \ldots\}$, $\mathbf{t}_k = \{t_{1,k}, t_{2,k}\}$, are operator equations of the first kind. Hence, their solution by a simple reduction method often turns out to be numerically unstable, while the representation of the wave field by a finite sum instead of a series is inapplicable in the near zone and, all the more, on the joining line where series in normal modes may even be divergent. The system can be regularized when it is separated in explicit form or by taking into account in the numerical scheme the behaviour of the solution at corner points, which determines the asymptotic form of the unknown $\mathbf{t}_k$ when $k \to \infty$ [3–5].

For example, for a known behaviour of $\mathbf{t}_k$

$$\mathbf{t}_k \sim \mathbf{c}d_k(\gamma), \quad k \to \infty \tag{2}$$

(in general, there may be several terms in (2) depending on the number of corner points or it may be necessary to take into account several terms of the expansion), system (1) can be reduced to an asymptotically equivalent stable system of the form

$$\sum_{k=1}^{N} a_{lk}\mathbf{t}_k + g_l\mathbf{c} = \mathbf{f}_l, \quad l = 1, 2, \ldots, N+1 \tag{3}$$

Here $\mathbf{c}$ is an unknown coefficient in the asymptotic representation (2) and $g_l = \Sigma_{k=N+1}^{\infty} a_{lk} d_k(\gamma)$. The factors $d_k(\gamma)$ depend on the index γ of the singularity in the stresses at the corner point, which governs the rate at which they decrease as $k \to \infty$.

Hence, the basis of this method of this method of regularization is obtaining asymptotic expansions of the type (2).

In the case of mixed problems, which can be reduced to Wiener–Hopf equations (contact problems [7] and diffraction by horizontal obstacles [8]), this approach turns out to be effective due to the use of explicit integral expressions for $\mathbf{t}_k$ in terms of a function with a known behaviour on the boundary, which enables an expansion $\mathbf{t}_k$ to be obtained as $k \to \infty$ as an asymptotic form of oscillating integrals [9]. In the case of composite waveguides of finite thickness with vertical boundaries an approach was proposed in [10] in which the property of generalized orthogonality of normal modes plays a similar role for obtaining the asymptotic form of $\mathbf{t}_k$ [1]. This enables $\mathbf{t}_k$ to be expressed in terms of integrals of the displacements and stresses at the vertical boundary, the nature of the behaviour of which at corner

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points of composite elastic solids is well known [11]. Moreover, the generalized orthogonality relation also enables one to use here the method of expansion in orthogonal polynomials with a weight carrying the required singularity at the corner points, so that the need to construct the asymptotic form of $\mathbf{t}_k$ drops out. Not only is stability of the solution of the system ensured here but also the possibility of carrying out a numerical analysis in the near zone up to the waveguide junction line.

Below we describe a form of this approach using the example of a composite elastic layer with a surface step.

We consider a composite elastic waveguide with a step consisting of two half-strips of thickness h_1 and h_2 ($h_1 \geqslant h_2$, $\Delta h = h_1 - h_2$ is the height of the step) with different constants of elasticity λ_n, μ_n and density ρ_n, $n = 1, 2$ (Fig. 1). On the junction line $x = 0$, $0 \leqslant z \leqslant h_2$ the conditions for rigid bonding (equality of the displacements and stresses) are satisfied, while the section of the end $h_2 \leqslant z \leqslant h_1$ and the upper surfaces $z = h_1$ when $x \leqslant 0$ and $z = h_2$ when $x \geqslant 0$ are stress-free; the lower surface $z = 0$ is rigidly fixed.

The harmonic wave field $\mathbf{u}e^{-i\omega t}$, $\mathbf{u} = \{u_x, u_z\} = \{u^{(1)}, u^{(2)}\}$ in the first (left) half-strip is made up of the specified field $\mathbf{u}_0e^{-i\omega t}$ and the field of reflected waves $\mathbf{u}_1e^{-i\omega t}$: $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, while in the second half-strip it consists of transmitted waves $\mathbf{u}_2e^{-i\omega t}$ (the harmonic factor $e^{-i\omega t}$ is henceforth omitted). When the waves are travelling from the right half-strip ($\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_2$ when $x > 0$) the proposed solution scheme is not changed in principle. When carrying out the calculations either an individual normal mode arriving from infinity

$$\mathbf{u}_0(x,z) = \mathbf{a}_{1,k}(z)e^{i\zeta_k x} \tag{4}$$

was specified as the field $\mathbf{u}_0$, or the field excited in an homogeneous layer of the normal lumped load $\sigma_z = p_0\delta(x - x_0)$, applied to the surface of the left half-strip ($x_0 < 0$)

$$\mathbf{u}_0(x,z) = ip_0\sum_{k=1}^{\infty}\mathbf{a}_{1,k}(x - x_0, z)e^{i\zeta_k|x - x_0|} \tag{5}$$

For the reflected and transmitted waves $\mathbf{u}_1$ and $\mathbf{u}_2$ and the corresponding fields of the stresses $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ ($\boldsymbol{\tau} = \{\sigma_{xx}, \tau_{xz}\}$), we also have an expansion in normal modes

$$\mathbf{u}_n(x,z) = \sum_{k=1}^{\infty} t_{n,k}\mathbf{a}_{n,k}(z)e^{\mp i\zeta_{n,k}x}, \quad \boldsymbol{\tau}_n(x,z) = \sum_{k=1}^{\infty} t_{n,k}\mathbf{b}_{n,k}(z)e^{\mp i\zeta_{n,k}x} \quad 0 \leqslant z \leqslant h_n, \ n = 1,2 \tag{6}$$

$$\mathbf{a}_{n,k} = \{\mathrm{sign}\, x i\zeta_{n,k}P_n(\zeta_{n,k}, z - h_n), \ \ R_n(\zeta_{n,k}, z - h_n)\} / \Delta_n'(\zeta_{n,k})$$

$$\mathbf{b}_{n,k}(z)e^{\mp i\zeta_{n,k}x} = T_n(\mathbf{a}_{n,k}(z)e^{\mp i\zeta_{n,k}x}), \quad T_n = \begin{Vmatrix} (\lambda_n + 2\mu_n)\dfrac{\partial}{\partial x} & 2\mu_n\dfrac{\partial}{\partial z} \\ \mu_n\dfrac{\partial}{\partial z} & \mu_n\dfrac{\partial}{\partial x} \end{Vmatrix}$$

Here and henceforth when $n = 1, x \leqslant 0$ we take the upper sign and when $n = 1, x \geqslant 0$ we take the lower sign.

The form of the functions $P_n(\alpha, z)/\Delta_n(\alpha)$ and $R_n(\alpha, z)/\Delta_n(\alpha)$, which occur in the second column of the symbol of Green's matrix $K(\alpha_1, \alpha_2)$, for the layer considered, bonded to the rigid base, is derived in [7]; the wave numbers $\pm\zeta_{n,k}$ ($\mathrm{Im}\zeta_{n,k} \geqslant 0$) are the poles of $K(\alpha_1, \alpha_2)$ (the zeros are $\Delta_n(\alpha)$,

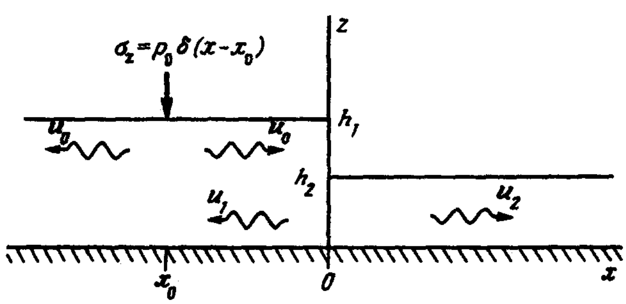

Fig. 1.

$\alpha = \sqrt{(\alpha_1^2 + \alpha_2^2)}$). The eigenvectors $\mathbf{b}_{n,k}$ are defined in terms of $\mathbf{a}_{n,k}$ using the stress operators T_n for the vertical area with normal parallel to the x axis.

The conditions on the vertical boundary $x = 0$ have the form

$$\mathbf{u}_0 + \mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\tau}_0 + \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2, \quad 0 \leqslant z \leqslant h_2 \tag{7}$$

$$\boldsymbol{\tau}_0 + \boldsymbol{\tau}_1 = 0, \quad h_2 \leqslant z \leqslant h_1 \tag{8}$$

$$(\boldsymbol{\tau}_0 = T_1\mathbf{u}_0)$$

The property of generalized orthogonality of normal modes [1]

$$(a_{n,m}^{(1)}, b_{n,j}^{(1)})_{h_n} - (a_{n,j}^{(2)}, b_{n,m}^{(2)})_{h_n} = 0, \quad m \neq j, \quad n = 1,2 \tag{9}$$

$$\left((f,g)_h = \int_0^h fg dz\right)$$

enables one to obtain an explicit solution in series of the auxiliary problem for a half-strip with opposite components of the displacement and stress vectors specified on the ends (the normal component of the displacements $u^{(1)}$ and the shears $\tau^{(2)}$ or vice versa).

Suppose, for example, that $u^{(1)}(0, z) = \upsilon(z)$, $\tau^{(2)}(0, z) = \sigma(z)$, where υ and σ are unknown functions in the interval $[0, h_1]$. It then follows from the boundary conditions

$$u_1^{(1)}(0,z) + u_0^{(1)}(0,z) = \upsilon(z), \quad \tau_1^{(2)}(0,z) + \tau_0^{(2)}(0,z) = \sigma(z), \quad 0 \leqslant z \leqslant h_1 \tag{10}$$

$$u_2^{(1)}(0,z) = \upsilon(z), \quad \tau_2^{(2)}(0,z) = \sigma(z), \quad 0 \leqslant z \leqslant h_2 \tag{11}$$

and relations (6) and (9), that

$$t_{1,k} = [(\upsilon - u_0^{(1)}, b_{1,k}^{(1)})_{h_1} - (\sigma - \tau_0^{(2)}, a_{1,k}^{(2)})_{h_1}]/d_{1,k} \tag{12}$$

$$t_{2,k} = [(\upsilon, b_{2,k}^{(1)})_{h_2} - (\sigma, a_{2,k}^{(2)})_{h_2}]/d_{2,k}$$

$$d_{n,k} = (a_{n,k}^{(1)}, b_{n,k}^{(1)})_{h_n} - (a_{n,k}^{(2)}, b_{n,k}^{(2)})_{h_n}, \quad n = 1,2$$

Note that the function $\sigma(z)$ is only unknown on the junction line $[0, h_2]$, and when $z \in [h_2, h_1]$ it follows from condition (8) that $\sigma(z) \equiv 0$.

The equations for finding υ and σ, in terms of which the required solution is expressed, by virtue of relations (6) and (12), are obtained from conditions (7) and (8) for the components $u^{(2)}$, $\tau^{(1)}$, which have not yet been used

$$\sum_{k=1}^{\infty}\left[t_{1,k}b_{1,k}^{(1)} - t_{2,k}\begin{Bmatrix} 0, & h_2 \leqslant z \leqslant h_1 \\ b_{2,k}^{(1)}, & 0 \leqslant z \leqslant h_2 \end{Bmatrix}\right] = -\tau_0^{(1)}, \quad 0 \leqslant z \leqslant h_1 \tag{13}$$

$$\sum_{k=1}^{\infty}\left[t_{1,k}a_{1,k}^{(2)} - t_{2,k}a_{2,k}^{(2)}\right] = -u_0^{(2)}, \quad 0 \leqslant z \leqslant h_2 \tag{14}$$

Using the general Bubnov–Galerkin scheme, Eqs (13) and (14) are projected onto the complete systems of coordinate functions $\{\varphi_{n,l}\}_{l=1}^{\infty}$, $n = 1, 2$ in the sections $[0, h_n]$. (We can take as the projectors $\varphi_{n,l}$ the Legendre polynomials $P_l(\xi_n)$, $\xi_n = 2z/h_n - 1 \in [-1, 1]$, $n = 1, 2$.) As a result, the partitioned elements a_{lk} of matrix A of system (1) and the components $f_{i,l}$ on the right-hand side $\mathbf{f} = \{\mathbf{f}_1, \mathbf{f}_2, \ldots\}$, $\mathbf{f}_l = \{f_{1,l}, f_{2,l}\}$ take the form

$$a_{lk} = \begin{Vmatrix} (\varphi_{1,l}, b_{1,k}^{(1)})_{h_1} & -(\varphi_{2,l}b_{2,k}^{(1)})_{h_2} \\ (\varphi_{2,l}, a_{1,k}^{(2)})_{h_2} & -(\varphi_{2,l}a_{2,k}^{(2)})_{h_2} \end{Vmatrix}$$

$$\mathbf{f}_l = \{-(\tau_0^{(1)}, \varphi_{1,l})_{h_1}, -(u_0^{(2)}, \varphi_{2,l})_{h_2}\}$$

Representation (12) for the unknowns $t_{n,k}$ in terms of the integrals of υ, σ enables us to construct the asymptotic form $t_{n,k}$ as $k \to \infty$, since the behaviour of the functions υ, σ at the points $z = 0, h_2, h_1$ which make the main contribution is easily obtained, starting from the results for the vertex of the composite elastic wedge [11]. Specifically for the waveguide geometry considered, the values of the stress singularity indicators γ_1, γ_2 at these points were specially calculated previously for the whole range of possible combinations of elastic properties of the half-strips.†

Another method of using this information to regularize the system is to expand the unknown functions $\upsilon(z)$, $\sigma(z)$ in Jacobi polynomials $P_m^{(\alpha,\beta)}(\xi)$ with a weight, which gives the required behaviour at these points

$$\upsilon(z) = \sum_{m=1}^{M} s_m^{(1)} \psi_m^{(1)}(z), \quad 0 \leqslant z \leqslant h_1 \tag{15}$$

$$\sigma(z) = \sum_{m=1}^{M} s_m^{(2)} \psi_m^{(2)}(z), \quad 0 \leqslant z \leqslant h_2$$

$$\begin{aligned} \psi_m^{(1)}(z) &= (1+\xi_1)^{1+\gamma_1} P_m^{(\gamma_1,0)}(\xi_1) \\ \psi_m^{(2)}(z) &= (1+\xi_2)^{\gamma_1}(1-\xi_2)^{\gamma_2} P_m^{(\gamma_1,\gamma_2)}(\xi_2) \end{aligned} \tag{16}$$

(since $\upsilon'(z)$ has a discontinuity at $z = h_2$, greater accuracy is obtained (but with more complicated calculations) by expanding $\upsilon(z)$ separately in each of the sections $[0, h_2]$, $[h_2, h_1]$ with a bonding condition at $z = h_2$)

$$\sum_{m=1}^{M} b_{lm} \mathbf{s}_m = \mathbf{f}_l - \mathbf{p}_l, \quad l = 1, 2, \ldots, M \tag{17}$$

$$b_{lm} = \sum_{k=1}^{\infty} a_{lk} r_{km}, \quad \mathbf{p}_l = \sum_{k=1}^{\infty} a_{lk} \mathbf{g}_k$$

to which system (1) reduces by substituting the relations

$$\mathbf{t}_k = \sum_{m=1}^{M} r_{km} \mathbf{s}_m + \mathbf{g}_k$$

$$r_{km} = \begin{Vmatrix} (b_{1,k}^{(1)} \psi_m^{(1)})_{h_1} / d_{1,k} - (a_{1,k}^{(2)}, \psi_m^{(2)})_{h_2} / d_{1,k} \\ (b_{2,k}^{(1)} \psi_m^{(1)})_{h_2} / d_{2,k} - (a_{2,k}^{(2)}, \psi_m^{(2)})_{h_2} / d_{2,k} \end{Vmatrix}$$

$$\mathbf{g}_k = \{[-(b_{1,k}^{(1)}, u_0^{(1)})_{h_1} + (a_{1,k}^{(2)}, \tau_0^{(2)})_{h_1}] / d_{1,k}, 0\} \tag{18}$$

which follow from (12) and (15).

System (17) has already been regularized, its solution is numerically stable, and to achieve the required accuracy a considerably lower dimensionality M is usually required than when using the stabilized system (2). A definite difficulty is the summation of the series when calculating the elements b_{lk} of the matrix and of the vector p_1, which, if necessary, can be overcome by standard methods of accelerating the convergence by taking into account the asymptotic forms of a_{lk}, r_{km}, $\mathbf{g}_k$ as $k \to \infty$.

Even better results can be obtained by a hybrid method, when the replacement (18) is made in system (1), beginning with a certain number $N + 1$, which leads to the hybrid system

$$\sum_{k=1}^{N} a_{lk} \mathbf{t}_k + \sum_{m=1}^{M} b_{lm} \mathbf{s}_m = \mathbf{f}_l - \mathbf{p}_l, \quad l = 1, 2, \ldots, N + M \tag{19}$$

with respect to the unknowns $\mathbf{t}_1, \ldots, \mathbf{t}_N$ and $\mathbf{s}_1, \ldots, \mathbf{s}_M$ (the series for b_{lm}, $\mathbf{p}_l$ begin here with $k = N + 1$). The second sum here plays the role of a stabilizing correction, which ensures numerical stability as N increases. The hybrid scheme (18), (19) can thereby be considered as another method of regularizing system (1) without the need to construct the asymptotic form $t_{n,k}$.

† GLUSHKOV, Ye. V. and GLUSHKOVA, N. V., The features of the solution at corner points of composite elastic waveguides. Deposited at the All-Union Institute of Scientific and Technical Information, N824-B91, 19 February 1991.

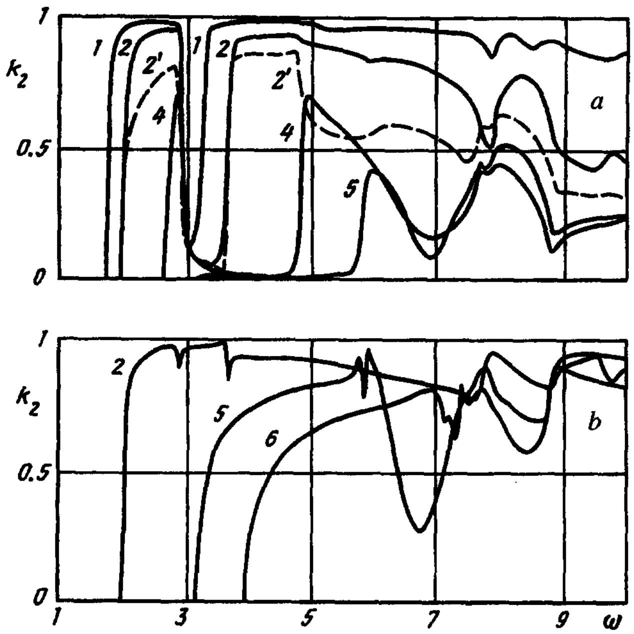

Fig. 2.

The effectiveness of the scheme can be increased even further if the oscillation of $\upsilon(z)$, $\sigma(z)$, due to the contribution of travelling modes (real $\zeta_{1,k}$), is not approximated by polynomials, only the contribution of terms which decrease exponentially with z, corresponding to complex $\zeta_{1,k}$, remaining, i.e. by taking

$$v(z) = u_0^{(1)}(0,z) + u_1^{(1)}(0,z) - \sum_{j=1}^{N} t_{1,j} a_1^{(1)}(z)$$

$$\sigma(z) = \tau_0^{(2)}(0,z) + \tau_1^{(2)}(0,z) - \sum_{j=1}^{N} t_{1,j} b_1^{(2)}(z)$$

(here N is greater than or equal to the number of real $\zeta_{1,k}$). This leads to a more complex linear relationship between $\mathbf{t}_k$ and $\mathbf{s}_m$ than (18), but promises a higher accuracy both for the coefficients $\mathbf{t}_k$ of the travelling modes, responsible for the energy balance, and for the stress intensity factors at the corner points, which are expressed in terms of $\mathbf{s}_m$.

As an example we show in Fig. 2 the results of calculations of the frequency dependence of the energy transmission coefficients $k_2 = E_2/E_0$ (the ratio of the energy of the passing waves $\bar{u}_2$ to the energy of the initial field $\bar{u}_0$) for a different height of the step Δh of a uniform waveguide with dimensionless parameters $h_1 = 1$, $h_2 = 1 - \Delta h$, $\rho_n = 1$, $\mu_n = 1$, a Poisson's ratio $\nu_n = 0.3$, $n = 1.2$ and also for a waveguide of different modulus with $\mu_2 = 0.5$ and $\Delta h = 0.2$ (the dashed curve 2′ in Fig. 2a). The case of diffraction of the first mode (4) ($k = 1$), arriving from the left from the thicker half-strip (a) and the opposite case of transmission from right to left, i.e. from the thin half-strip to the thick one (b) is shown. The results for $\Delta h = 0.1, 0.2, \ldots, 0.6$ are represented by lines 1–6, respectively.

For Fig. 2(a) a characteristic feature is the sharp reduction in the transmission coefficient in the frequency band $2.94 < \omega < 2.94/h_2$, which is due to the fact that for an elastic layer with a single fixed boundary the first travelling mode appears for a value of the dimensionless wave number $\varkappa = \omega h/\upsilon_S = 1.57$ ($\upsilon_s = \sqrt{(\mu/\rho)}$), the second for $\varkappa = 2.94$ (with a range of the backward wave of 2.88 – 2.94) and, subsequent ones with $\varkappa = 4.71, 7.85, \ldots$. A second mode appears from the left when $\omega > 2.94$, whereas only one travelling wave is excited from the right when $\omega < 2.94/h_2$ and leads to strong reflection of the signal in this band (Fig. 2a). A stronger reflection as a whole for high steps ($\Delta h > 0.5$) is also obvious due to the larger number of reflected travelling modes, which carry energy to the left, than passing modes, whereas the transmission of energy from right to left (Fig. 2b) depends only slightly on the height of the step. However, a partial-cutoff effect is also observed here ($\omega \approx 6.8$, $\Delta h = 0.5$), but it is more likely related to the specific structure of the streamlines of the energy flow, averaged over an oscillation period, which is characterized by the occurrence of energy vortices in the region of the junction line, which cover a considerable part of the cross-section of the waveguide up to complete blocking at the cutoff frequency [8].

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